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Quantum eigenfunctions in terms of periodic orbits of chaotic systems

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Abstract. A resummed formula for the Wigner function, corresponding to an eigenfunction of a chaotic system, in terms of periodic orbits, is developed. The infinite sum over periodic orbits is effectively truncated with the help of an extension of a method that was applied to the spectral determinant by Berry and Keating. In principle, the formula enables the computation of eigenstates and the probability density of wavefunctions from classical periodic orbits. The conditions for appearance of 'scars' are discussed.

1. Introduction

The quantal behaviour of systems that are chaotic in the classical limit has been investigated extensively in recent years [1–6]. Many of these studies were inspired by Gutzwiller's trace formula that expresses the density of states of chaotic system, in terms of a sum over its classical periodic orbits [7]. The main problem is that this sum is not absolutely convergent for any real energy. For this reason some interpretation of these series is required. Several resummation methods were recently introduced [8–15]. The main conclusion from these studies is that the sum over periodic orbits should be effectively truncated after a finite number of pseudo-orbits. The period length of the longest orbit that should be taken into account for the calculation of the eigenenergies is inversely proportional to the mean energy spacing. A different and more complicated problem is to find an efficient method to sum this exponentially large number (in orbit length) of terms. There are some suggestions for such methods that are not related to the present work [16]. Since the spectrum is related to the corresponding eigenfunctions, one would like to develop a method that will enable us to express the eigenstates of the system in terms of periodic orbits, in the semiclassical limit. This is the main subject of the paper.

For some chaotic systems it was found that some eigenstates are strongly peaked near periodic orbits that are unstable. These imprints of the periodic orbits were termed as 'scars' by Heller [17, 18]. Heller gave a heuristic argument for the existence of scars, and an estimate of their magnitude. The scar phenomenon was investigated numerically [19–26] and experimentally [27, 28] for a variety of systems. It was analysed in the framework of periodic orbit theory by Bogomolny [29] in the configuration space, and by Berry [30, 31] in phase space. In order to get meaningful results they calculated the corresponding function, namely the Green function and the spectral Wigner function, smeared over some energy range. Therefore the relation of these functions to eigenstates corresponding to a well defined eigenenergy is not transparent.

Currently, the wavefunctions of chaotic systems are also investigated [32] using time domain techniques [33]. In this approach the eigenstates are constructed by direct Fourier transform of the Van-Vleck propagator.

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In the present paper an *approximate semiclassical formula* for the eigenstates is developed. It is expected to be correct in the leading order in Planck's constant $\hbar$. The starting point is a semiclassical formula for the resolvent operator which is almost identical to the formula of the spectral Wigner function developed by Berry [30] for finite energy smearing. In the limit of zero smearing the meaning of this formula is not clear as is the case for Gutzwiller's trace formula. The technique that was developed by Berry and Keating [8] in order to obtain a meaningful formula for the spectral determinant will be applied extensively in the present work.

The main result of the paper is a formula for the Wigner function corresponding to an eigenstate in terms of effectively finite number of periodic orbits. This expression is integrated over the momenta to give the corresponding probability density. In addition an approximate expression for the strength of scars in terms of the periodic orbits is obtained.

The outline of the paper is as follows. The resummation technique that was used by Berry and Keating [8] for the spectral determinant is summarized in section 2. Derivation of a semiclassical formula for individual wavefunctions in the Wigner–Weyl representation is presented in section 3, and an analogous formula for the semiclassical probability density of the eigenstates is derived in section 4. In section 5, the results of the previous sections are used in order to investigate the appearance of scars in individual wavefunctions. The results as well as some other related problems are discussed in section 6.

2. The spectral determinant

The semiclassical quantization rule for chaotic systems is usually formulated in terms of the dynamical zeta function $\zeta_s(E)$, also called the Selberg zeta function [8,9]. This function is defined as a product over primitive periodic orbits,

$$\zeta_s(E) = \prod_{\text{ppo}} \prod_j \left\{ 1 - \exp\left[\frac{\mathrm{i}}{\hbar} S_p(E) - \mathrm{i}\gamma_p - \left(\frac{1}{2} + j \right) u_p \right] \right\} \tag{2.1}$$

where the subscript p denotes a primitive periodic orbit, $S_p(E)$ is its corresponding action, γ_p is a phase determined by the focusing paths close to p, and u_p is the instability exponent. The eigenenergies of the system are the zeros of $\zeta_s(E)$, thus one may formulate the semiclassical quantization rule for chaotic systems to be

$$\zeta_s(E) = 0 . \tag{2.2}$$

In general, $\zeta_s(E)$ is not a real function for real values of E. Nevertheless, multiplying this function by a phase factor $\mathrm{e}^{-\mathrm{i}\pi\bar{N}(E)}$ where $\bar{N}(E)$ is the mean energy staircase, gives the semiclassical approximation of the spectral determinant which is a real function for real values of E [9]. Hence, it is convenient from now on to study the spectral determinant,

$$\Delta(E) = \mathrm{e}^{-\mathrm{i}\pi\bar{N}(E)} \zeta_s(E) . \tag{2.3}$$

Although there are some indications that one may obtain a good approximation for the lower zeros of $\Delta(E)$ by considering the real part of this function with the Euler product (2.1) truncated at some point [34], usually, resummation techniques are based on expressing $\Delta(E)$ as a Dirichlet sum over pseudo-orbits,

$$\Delta(E) = \sum_{\mu} c_\mu \mathrm{e}^{-\mathrm{i}\pi\bar{N}(E) + (\mathrm{i}/\hbar) S_\mu} . \tag{2.4}$$

Pseudo-orbits are linear combination of periodic orbits and may be obtained by expanding the Euler product (2.1) and collecting the terms according to some rule. Here, the pseudo-orbits will be ordered according to their period length. The following discussion regarding the calculation of the pseudo-orbits' amplitudes c_μ will be confined, for simplicity, to the case of two degrees of freedom. Nevertheless, the structure of (2.4) is general and also holds to higher dimensions.

Applying the Euler identity,

$$\prod_{j=0}^{\infty}(1-ax^j)=\sum_{m=0}^{\infty}\frac{(-1)^m a^m x^{m(m-3)/4}}{(x^{-\frac{1}{2}}-x^{\frac{1}{2}})(x^{-1}-x^1)\cdots(x^{-m/2}-x^{m/2})} \tag{2.5}$$

to the product of the dynamical zeta function (2.1), with the identification $x = \mathrm{e}^{-u_p}$ and $a = \exp((\mathrm{i}/\hbar)S_p(E) - \mathrm{i}\gamma_p - \frac{1}{2}u_p)$ yields

$$\zeta_s(E)=\prod_{\text{ppo}}\sum_{m=0}^{\infty}\frac{(-1)^m\,\mathrm{e}^{-\frac{1}{4}m(m-1)u_p}\mathrm{e}^{\mathrm{i}m(S_p/\hbar-\gamma_p)}}{\sqrt{\prod_{k=1}^{m}\det(M_p^k-I)}} \tag{2.6}$$

where the $m = 0$ term is unity, M_p is the monodromy matrix of the primitive periodic orbit p and $\mathrm{e}^{\pm u_p}$ are its eigenvalues. The phase factor suggests interpreting m as a repetition number. Expanding the product (2.6) and collecting terms so that the sum of the periods of all the orbits which construct pseudo-orbit μ equals $\mathcal{T}_\mu$ gives for systems with two degrees of freedom the result (2.4), where the amplitude c_μ of the pseudo-orbit μ is

$$c_\mu=\prod_{\sum T_p m_p=\mathcal{T}_\mu}\frac{(-1)^{m_p}\,\mathrm{e}^{-\frac{1}{4}m_p(m_p-1)u_p}}{\sqrt{\prod_{k=1}^{m_p}\det(M_p^k-I)}}\,\mathrm{e}^{-\mathrm{i}m_p\gamma_p} \tag{2.7}$$

with T_p the period the primitive periodic orbit p, and m_p is the number of its repetitions. The action of the pseudo-orbit $\mathcal{S}_\mu$ is a linear combination of the actions of the individual orbits,

$$\mathcal{S}_\mu=\sum_p m_p S_p\,. \tag{2.8}$$

The relation between the period of the pseudo-orbit $\mathcal{T}_\mu$ and its corresponding action $\mathcal{S}_\mu$ is clearly $\mathcal{T}_\mu = \partial\mathcal{S}_\mu/\partial E$.

For long orbits and pseudo-orbits of ergodic systems $\mathcal{T}_\mu$ and $\mathcal{S}_\mu$ are proportional and satisfy the relation [35],

$$\mathcal{S}_\mu=\frac{D\mathcal{T}_\mu\Omega(E)}{\Omega'(E)} \tag{2.9}$$

where D is the number of degrees of freedom of the system, $\Omega(E)$ the classical phase-space volume with energy less than E,

$$\Omega(E)=\iint \mathrm{d}p\,\mathrm{d}q\,\Theta[E-\mathcal{H}(q,p)] \tag{2.10}$$

and $\Omega'(E)$ is its derivative with respect to the energy. In the above equation, Θ is the unit step function.

The exponential growth in the number of periodic orbits as their period increases implies that the formal sum of (2.4) is not absolutely convergent [10, 36]. Convergence of this formula is obtained for a complex value of $1/\hbar$ satisfying [8]

$$\mathrm{Im}\frac{1}{\hbar} > \frac{\lambda(E)\Omega'(E)}{2D\Omega(E)} \tag{2.11}$$

where $\lambda(E)$ is the metric entropy, which was assumed to be equal to the topological entropy. Convergence of the semiclassical expansion for complex values of the energy was discussed by Eckhardt and Aurell [36].

The analytic continuation of (2.4) into the real axis of $1/\hbar$ is based on two important relations. The first one is the exact functional equation,

$$\Delta(E,\hbar) = \Delta(E,-\hbar) \tag{2.12}$$

which holds even for systems without time-reversal symmetry. This is due to the fact that changing $\hbar$ to $-\hbar$ transforms the Hamiltonian into its complex conjugate leaving its eigenvalues unchanged. The second relation is

$$c_\mu \mathrm{e}^{-\mathrm{i}\pi\bar{N}(E)+(\mathrm{i}/\hbar)S_\mu} \to \left[c_\mu \mathrm{e}^{-\mathrm{i}\pi\bar{N}(E)+(\mathrm{i}/\hbar)S_\mu}\right]^* \qquad \text{if} \quad \hbar \to -\hbar\,. \tag{2.13}$$

This relation follows from the behaviour of the time-dependent Schrödinger equation in the semiclassical limit, under $\hbar$ reversal [8].

Imposing the exact functional equation (2.12) on the semiclassical approximation and using relation (2.13) enables one to derive an asymptotic series which corresponds to analytical continuation into the real axis of $1/\hbar$. The first term of this expansion is analogous to the main sum of the Riemann–Siegel formula for the Riemann zeta function [10], with the sharp cut-off smoothed by the complementary error function. For ergodic systems the centre of this smoothing is at the pseudo-orbit μ^* whose period is

$$T_{\mu^*} = \pi\hbar\bar{d}(E) \tag{2.14}$$

where $\bar{d}(E) \approx \Omega'(E)/(2\pi\hbar)^D$ is the semiclassical smoothed level density. The high-order terms of this asymptotic expansion pick their contributions from pseudo-orbits located near the smoothing centre. Numerical tests of this expansion for the Riemann zeta function [8] have shown that the main sum alone already gives very good results. The approximate expression for the spectral determinant corresponding to the main sum of the Riemann–Siegel formula is,

$$\Delta(E) \approx \mathrm{Re}\sum_\mu c_\mu \mathrm{e}^{-\mathrm{i}\pi\bar{N}(E)+(\mathrm{i}/\hbar)S_\mu}\mathrm{Erfc}\left\{\frac{\xi(\mu,\hbar,E)}{B(K,\hbar,E)\sqrt{2\hbar}}\right\} \tag{2.15}$$

where

$$\xi(\mu,\hbar,E) \approx S_\mu - \frac{D\Omega(E)}{2(2\pi\hbar)^{D-1}} \tag{2.16}$$

and

$$B(K,\hbar,E)^2 \approx K^2 + \mathrm{i}\frac{D(D-1)\Omega(E)}{2(2\pi\hbar)^{D-1}}\,. \tag{2.17}$$

The smoothing centre (2.14) is found from the equation $\xi(\mu^*, \hbar, E) = 0$ using the non-trivial relation (2.9) which holds for ergodic systems. K is a free fine-tuning parameter, chosen to satisfy

$$0 < K \ll \frac{D\Omega(E)}{\Omega'(E)}\sqrt{\frac{\hbar}{2}}\pi\bar{d}(E).$$

It is assumed here that $D(D-1)\Omega(E)/2(2\pi\hbar)^{D-1} \ll K^2$. For small $\hbar$ or large E these requirements are consistent. Thus, the pseudo-orbits which contribute effectively to the main sum (2.15) are those for which

$$\mathcal{T}_\mu < \mathcal{T}_{\mu^*} + K\frac{\Omega'(E)}{D\Omega(E)}\sqrt{2\hbar}. \tag{2.18}$$

The higher terms of the asymptotic expansion get their contribution from pseudo-orbits with period $\mathcal{T}_\mu$ in a small interval near $\mathcal{T}_{\mu^*}$, namely

$$\mathcal{T}_{\mu^*} - K\frac{\Omega'(E)}{D\Omega(E)}\sqrt{2\hbar} < \mathcal{T}_\mu < \mathcal{T}_{\mu^*} + K\frac{\Omega'(E)}{D\Omega(E)}\sqrt{2\hbar}. \tag{2.19}$$

The minimal width of the smoothing region and of the correction terms is obtained for $K^2 = D(D-1)\Omega(E)/2(2\pi\hbar)^{D-1}$.

3. Semiclassical Wigner functions

In this section, the semiclassical formula for the Wigner function, corresponding to an eigenfunction, is derived. The discussion will be divided into two stages. First, a semiclassical expression for the resolvent operator in its Wigner–Weyl representation is derived for complex values of $1/\hbar$. The starting point of this analysis is a semiclassical expression for the resolvent operator in the Wigner–Weyl representation, which is similar to the scar formula obtained by Berry [30, 31]. Then, the resolvent operator is analytically continued into the real axis of $1/\hbar$, and the Wigner functions are identified from the residue of the different poles of the resulting expression.

3.1. The semiclassical resolvent operator

Let $\mathcal{H}(\boldsymbol{x})$ be the Hamiltonian of a chaotic system with D degrees of freedom where $\boldsymbol{x} = (q_1, \ldots q_D, p_1, \ldots p_D)$ are the coordinates and momenta. The resolvent operator is

$$\hat{\mathcal{R}} = \frac{1}{E + \mathrm{i}\varepsilon - \hat{\mathcal{H}}} \tag{3.1}$$

which may also be written as a time integral over the propagator,

$$\hat{\mathcal{R}} = \frac{-\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{(\mathrm{i}/\hbar)(E-\hat{\mathcal{H}})t - \varepsilon t/\hbar} \tag{3.2}$$

We define the resolvent Wigner function to be the Weyl transform of this operator. Thus,

$$W(\boldsymbol{x}; E) = \int \mathrm{d}q'\, \mathrm{e}^{-(\mathrm{i}/\hbar)pq'} \langle q + \tfrac{1}{2}q' \mid \hat{\mathcal{R}} \mid q - \tfrac{1}{2}q' \rangle = h^D \sum_\alpha \frac{W_\alpha(\boldsymbol{x})}{E + \mathrm{i}\varepsilon - E_\alpha} \tag{3.3}$$

where $W_\alpha(x)$ is the Wigner function of the eigenstate ψ_α, and E_α is its corresponding eigenenergy. The main goal of this work is to develop approximate expressions for the $W_\alpha(x)$.

Using the time integral representation of the resolvent (3.2), one may express the resolvent Wigner function (3.3) in terms of the Wigner propagator,

$$K_W(x;t) = \int \mathrm{d}q' \, \mathrm{e}^{-(\mathrm{i}/\hbar)pq'} \left\langle q + \tfrac{1}{2}q' \middle| \mathrm{e}^{-\mathrm{i}\hat{\mathcal{H}}t/\hbar} \middle| q - \tfrac{1}{2}q' \right\rangle \tag{3.4}$$

as

$$W(x;E) = \frac{-\mathrm{i}}{\hbar} \int_0^\infty \mathrm{d}t \, \mathrm{e}^{(\mathrm{i}/\hbar)Et - \varepsilon t/\hbar} K_W(x;t) \,. \tag{3.5}$$

The semiclassical analysis continues by introducing the semiclassical approximation for the propagator (3.4) as a sum over classical trajectories connecting $q - \frac{1}{2}q'$ with $q + \frac{1}{2}q'$ in time t, and performing the integrals of (3.4) and (3.5) in the stationary phase approximation. This procedure is identical to the one introduced by Berry [30]. The result one obtains for the resolvent Wigner function in the semiclassical approximation is

$$W(x;E) = \frac{1}{E + \mathrm{i}\varepsilon - \mathcal{H}(x)} + \sum_p W^p_{\mathrm{scar}}(x;E,\varepsilon) \,. \tag{3.6}$$

It contains contributions of two sorts: the first term originates from the zero-length orbits, while the other term is the contribution of closed periodic orbits (primitive and repeated) as well as other orbits in their vicinity. Each one of the latter has the form

$$W^p_{\mathrm{scar}}(x;E,\varepsilon) = -\mathrm{i}\pi \, \mathrm{e}^{-\varepsilon T_p/\hbar} \frac{2^D}{\sqrt{\det(M_p + I)}} A^p(x,E) \times \exp\left\{ \mathrm{i} \left(\frac{1}{\hbar} \left[S_p + \bar{X} J \frac{M_p - I}{M_p + I} X \right] - \gamma_p \right) \right\} \tag{3.7}$$

where

$$A^p(x,E) = \frac{2}{\hbar |(\ddot{x} \wedge \dot{x})/\hbar|^{\frac{1}{3}}} \mathrm{Ai}\left(\frac{2[\mathcal{H}(x) - E]}{(\hbar^2 \ddot{x} \wedge \dot{x})^{\frac{1}{3}}} \right) . \tag{3.8}$$

The period of the periodic trajectory p is T_p, while M_p is the corresponding monodromy matrix, S_p is its action and γ_p is the Maslov phase. The other terms of (3.7) are expressed in terms of the following canonical variables:

$$\mathcal{H}, \qquad t, \qquad \text{and} \qquad X(Q,P) = (Q_1, \ldots Q_{D-1}, P_1, \ldots P_{D-1}) \tag{3.9}$$

where $\mathcal{H}$ is the Hamiltonian, X are the $2D-2$ coordinates on the Poincaré surface of section, and t is the time along the periodic orbit measured from the surface of section. The exponent term in (3.7) forms a structure of quadratic fringes as x moves off the closed orbit. In this quadratic term I is the unit matrix, while J is the unit symplectic matrix,

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} . \tag{3.10}$$

The Airy term $A^p(x,E)$ describes the pattern of fringes as x moves off the energy surface. This factor involves the phase-space velocity $\dot{x}$ and acceleration $\ddot{x}$ at point x on the orbit.

Note that the differences between formula (3.6) and the corresponding one derived by Berry are due to the fact that here we consider the resolvent operator and not only its imaginary part. In the derivation of (3.6) it was assumed that $|\boldsymbol{X}|$ and $|\mathcal{H}(\boldsymbol{x}) - E|$ are small, namely that the closed orbits are in the vicinity of $\boldsymbol{x}$. Contributions from more distant orbits are effectively averaged to zero.

As stated by Berry [30], the convergence of formula (3.6) as $\varepsilon \to 0$ is doubtful, and therefore one cannot resolve individual W_α from it. However, convergence of (3.6) may also be achieved if, instead of looking at complex values of the energy, one considers complex values of $1/\hbar$ satisfying condition (2.11). Thus henceforth ε will be considered infinitesimal (and therefore it will be omitted from most of the the following formulae), while $1/\hbar$ will contain an imaginary part which ensures the convergence of (3.6).

It turns out that in order to identify the $W_\alpha(\boldsymbol{x})$, one needs to sum up the contributions from the repeated orbits. Doing it for the general case involves complicated expressions. Therefore, for simplicity, the following discussion will be restricted to systems with two degrees of freedom ($D = 2$) which are hyperbolic. Thus, it will be assumed that the eigenvalues of the monodromy matrices M_p are $\mathrm{e}^{\pm u_p}$. Generalization of this analysis to some other cases is straightforward.

To sum up the contributions from all repeated orbits, one has to modify (3.7) for W^p_{scar}, and to write it as a sum over powers of e^{-u_p}. Starting from the amplitude, it clearly satisfies

$$\frac{1}{\sqrt{\det(M_p + I)}} = \frac{1}{2\cosh(u_p/2)} = \sum_{k=0}^{\infty} (-1)^k \mathrm{e}^{-u_p\left(\frac{1}{2}+k\right)} . \tag{3.11}$$

It is somewhat more complicated to expand the exponent term in (3.7). Let us introduce the eigenvectors of the monodromy matrix, $\boldsymbol{v}_p^+(\boldsymbol{x})$ and $\boldsymbol{v}_p^-(\boldsymbol{x})$, corresponding to the eigenvalues e^{+u_p} and e^{-u_p} respectively. Obviously, these vectors depend on their position along the periodic orbits, since M_p does. Yet, the corresponding eigenvalues do not depend on this position. The eigenvectors $\boldsymbol{v}_p^{\pm}(\boldsymbol{x})$ also build the matrix which diagonalizes the monodromy matrix. It may be represented as

$$M_p = \frac{1}{\boldsymbol{v}_p^+ \wedge \boldsymbol{v}_p^-} \begin{pmatrix} v_{p1}^+ & v_{p1}^- \\ v_{p2}^+ & v_{p2}^- \end{pmatrix} \begin{pmatrix} \mathrm{e}^{u_p} & 0 \\ 0 & \mathrm{e}^{-u_p} \end{pmatrix} \begin{pmatrix} v_{p2}^- & -v_{p1}^- \\ -v_{p2}^+ & v_{p1}^+ \end{pmatrix} \tag{3.12}$$

where $\{v_{pi}^{\pm}\}_{i=1,2}$ are the components of the eigenvectors $\boldsymbol{v}_p^{\pm}(\boldsymbol{x})$, and $\boldsymbol{v}_p^+ \wedge \boldsymbol{v}_p^-$ is the symplectic area of the parallelogram defined by these vectors. It is shown in appendix A that using the above definitions one can write the quadratic term in (3.7) as

$$\tilde{\boldsymbol{X}} J \frac{M_p - I}{M_p + I} \boldsymbol{X} = \tanh\left(\frac{u_p}{2}\right) \tilde{\boldsymbol{X}} \boldsymbol{R}_p(\boldsymbol{x}) \boldsymbol{X} \tag{3.13}$$

where the matrix $\boldsymbol{R}_p(\boldsymbol{x})$ is

$$\boldsymbol{R}_p(\boldsymbol{x}) = \frac{1}{R_p^+ - R_p^-} \begin{pmatrix} -2R_p^+ R_p^- & R_p^+ + R_p^- \\ R_p^+ + R_p^- & -2 \end{pmatrix} \tag{3.14}$$

and

$$R_p^{\pm} = \frac{v_{p2}^{\pm}}{v_{p1}^{\pm}} . \tag{3.15}$$

When the instability exponent u_p is large, $z = e^{-u_p}$ is a small parameter and $\tanh(u_p/2)$ is almost independent of u_p and very close to 1. Thus the quadratic term in the exponent of

(3.7) can be approximated by (3.13) with $\tanh(u_p/2) \approx 1$. An exact expression is obtained by expanding the exponent term around $z = 0$. This expansion takes the form

$$e^{(i/\hbar)\tilde{X}J(M_p-I)/(M_p+I)X} = \sum_{l=0}^{\infty} f_p^{(l)} e^{-u_p l} e^{(i/\hbar)\tilde{X}R_pX} \tag{3.16}$$

where the functions $f_p^{(l)}$ are

$$f_p^{(l)} = e^{-(i/\hbar)\tilde{X}R_pX} \frac{1}{l!}\left(\frac{\partial}{\partial z}\right)^l e^{(i/\hbar)(1-z)/(1+z)\tilde{X}R_pX}\Big|_{z=0}. \tag{3.17}$$

Combining (3.11) and (3.16) together with (3.7) we obtain the required new form of W_{scar}^p,

$$W_{\text{scar}}^p(x; E) = -4\pi i A^p(x, E) \sum_{n=0}^{\infty} g_p^{(n)} e^{-u_p(\frac{1}{2}+n)} e^{(i/\hbar)\left(S_p+\tilde{X}R_pX\right)-i\gamma_p} \tag{3.18}$$

where $g_p^{(n)}$ are polynomials of the variable

$$b = b(x) = \frac{i}{\hbar}\tilde{X}R_pX \tag{3.19}$$

defined along the primitive periodic orbit p. In terms of $f_p^{(l)}$ these functions are given by

$$g_p^{(n)}(b) = \sum_{l=0}^{n} (-1)^{n-l} f_p^{(l)}. \tag{3.20}$$

The first few of them are

$$g_p^{(0)}(b) = 1 \qquad g_p^{(1)}(b) = -1 - 2b \qquad g_p^{(2)}(b) = 1 + 4b + 2b^2. \tag{3.21}$$

To see that now everything is prepared in order to sum up the repetitions of the primitive periodic orbits, consider the primitive periodic orbit p and the orbit which consists of r repetitions of p. Let $W_{\text{scar}}^{p,r}$ be the contribution of these repeated orbits to (3.6). One may obtain $W_{\text{scar}}^{p,r}$ from (3.18) for W_{scar}^p simply by replacing S_p, u_p, and γ_p by these quantities multiplied by r. Note that the Airy factor (3.8), as well as the matrix R_p and therefore the functions $g_p^{(n)}$, do not depend the number of repetitions r. Hence, the summation over the repetitions of (3.18) becomes trivial, since it is a simple geometric series. Let $W^p(x; E)$ be the contribution from the primitive periodic orbit p and all its repetitions; thus $W^p(x; E) = \sum_{r=1}^{\infty} W_{\text{scar}}^{p,r}$. It is straightforward to verify that

$$W^p(x; E) = -4\pi i A^p(x, E) \sum_{n=0}^{\infty} \frac{g_p^{(n)} e^{-u_p(\frac{1}{2}+n)} e^{(i/\hbar)\left(S_p+\tilde{X}R_pX\right)-i\gamma_p}}{1 - e^{-u_p(\frac{1}{2}+n)} e^{(i/\hbar)S_p - i\gamma_p}}. \tag{3.22}$$

The expression for the resolvent Wigner function in terms of W^p is clearly

$$W(x; E) = W_0(x; E) + \sum_p W^p(x; E) \tag{3.23}$$

with

$$W_0(x; E) = \frac{1}{E - \mathcal{H}(x)} \tag{3.24}$$

where now the summation is only over primitive periodic orbits, since their repetitions are already taken into account. The common denominator of all the $W^p(x; E)$ is $\zeta_s(E)$ of (2.1). Multiplying the denominator and the numerator by $e^{-i\pi\bar{N}(E)}$ enables one to write the resolvent Wigner function in a form which reveals its poles,

$$W(x; E) = W_0(x; E) + \frac{4\pi}{\Delta(E)} \sum_{\text{ppo}} \sum_{n} \Delta^{(p,n)}(E) A^p(x, E) g_p^{(n)}(x)\, e^{(i/\hbar)\bar{X}R_pX} \tag{3.25}$$

where $\Delta^{(p,n)}(E)$ is a function similar to the spectral determinant (2.3), except that the factor $\left(1 - e^{-u_p(\frac{1}{2}+n)}\, e^{(i/\hbar)S_p - i\gamma_p}\right)$ is replaced by $-ie^{-u_p(\frac{1}{2}+n)}e^{(i/\hbar)S_p - i\gamma_p}$. Thus,

$$\begin{aligned}\Delta^{(p,n)}(E) = -ie^{-i\pi\bar{N}(E)} &\prod_{p' \neq p} \prod_{j} \left\{1 - \exp\left[\frac{i}{\hbar}S_{p'}(E) - i\gamma_{p'} - \left(\frac{1}{2}+j\right)u_{p'}\right]\right\} \\ &\times \prod_{j \neq n} \left\{1 - \exp\left[\frac{i}{\hbar}S_p(E) - i\gamma_p - \left(\frac{1}{2}+j\right)u_p\right]\right\} \\ &\times \exp\left[\frac{i}{\hbar}S_p(E) - i\gamma_p - \left(\frac{1}{2}+n\right)u_p\right].\end{aligned} \tag{3.26}$$

Note that due to the imaginary part of $1/\hbar$, all sums and products in (3.25) converge. However, this formula cannot be used for real values of $1/\hbar$, and therefore analytic continuation is needed.

3.2. Analytic continuation

A proper expression for (3.25) at real values of $1/\hbar$ will be obtained in what follows, using the same powerful method of analytic continuation in the variable $1/\hbar$ [8] described in section 2. The key for such resummation is the exact functional equation satisfied by the Wigner functions for real $\hbar$, namely,

$$\bar{W}_\alpha(x, \hbar) = \bar{W}_\alpha(x, -\hbar) \tag{3.27}$$

where

$$\bar{W}_\alpha(x, \hbar) = h^D W_\alpha(x) \tag{3.28}$$

and $W_\alpha(x)$ is the Wigner function corresponding to the state ψ_α. Like (2.12), the relation (3.27) also holds for systems without time-reversal symmetry. This is shown in appendix B.

The present form of the resolvent Wigner function (3.25) is still inadequate for our purpose, and it is necessary to convert it into a more appropriate one. Starting from the functions $\Delta^{(p,n)}(E)$ which are represented by (3.26) as Euler products, we convert them into Dirichlet sums over pseudo-orbits,

$$\Delta^{(p,n)}(E) = -i \sum_{\mu} c_\mu^{(p,n)} e^{-i\pi\bar{N}(E) + (i/\hbar)S_{\mu,p}} \tag{3.29}$$

where

$$S_{\mu,p} = S_\mu + S_p \tag{3.30}$$

with $\mathcal{S}_\mu$ given by (2.8), S_p is the action of the primitive periodic orbit p, and $c_\mu^{(p,n)}$ are amplitudes which are different from those of the spectral determinant (2.7). The construction of these amplitudes is discussed in appendix C.

Next, it is convenient to remove the explicit dependence of $g_p^{(n)}(b)$ on $\hbar$. This is done by introducing a dummy variable η which multiplies the quadratic term of the exponent of (3.25), namely by the replacement

$$\tilde{\boldsymbol{X}}\boldsymbol{R}_p\boldsymbol{X} \to \eta\tilde{\boldsymbol{X}}\boldsymbol{R}_p\boldsymbol{X}\,. \tag{3.31}$$

The functions $g_p^{(n)}$ are now expressed as polynomials of derivatives with respect to η. For example,

$$g_p^{(0)}(\partial_\eta) = 1 \qquad g_p^{(1)}(\partial_\eta) = -1 - 2\partial_\eta \qquad g_p^{(2)}(\partial_\eta) = 1 + 4\partial_\eta + 2(\partial_\eta)^2\,. \tag{3.32}$$

After taking the derivatives, one has to set η to 1. Defining also

$$\bar{\mathcal{S}}_\mu^p(\boldsymbol{x}, \eta) = \mathcal{S}_{\mu,p} + \eta\tilde{\boldsymbol{X}}\boldsymbol{R}_p\boldsymbol{X} \tag{3.33}$$

and substituting (3.29), (3.31) and (3.33) into (3.25) yields the resolvent Wigner function,

$$W(\boldsymbol{x}; E) = W_0(\boldsymbol{x}; E) + \frac{\mathcal{N}(\boldsymbol{x}; 1/\hbar)}{\Delta(E)} \tag{3.34}$$

with $\mathcal{N}(\boldsymbol{x}; 1/\hbar)$ as a triple sum of the form

$$\mathcal{N}\left(\boldsymbol{x}; \frac{1}{\hbar}\right) = -4\pi\mathrm{i}\sum_{p,n,\mu} A^p(\boldsymbol{x}, E)c_\mu^{(p,n)}g_p^{(n)}(\partial_\eta)\mathrm{e}^{-\mathrm{i}\pi\bar{N}+(\mathrm{i}/\hbar)\bar{\mathcal{S}}_\mu^p(\boldsymbol{x},\eta)} \tag{3.35}$$

where p runs over primitive periodic orbits, μ corresponds to a sum over pseudo-orbits resulting from the expression (3.29), while the sum over n takes into account the corrections which originate from the finite value of the instability exponents.

Equation (3.34) is now in a proper form for the analytic continuation. Note that $\mathcal{H}(\boldsymbol{x})$ is independent of $\hbar$, therefore only the second term of (3.34) requires analytic continuation. This will be done for the denominator and the numerator separately. The denominator is the spectral determinant, and its analytic continuation was already described in section 2, following [8]. Therefore we are left only with the problem of the continuation of the triple sum of (3.35). Following the analysis of the spectral determinant (section 2 and [8]) it will be shown that this sum alone is invariant under $\hbar$ reversal. For this purpose it is instructive to introduce the regularized form of the spectral determinant, where each factor $(E - E_\alpha)$ is replaced by $C(E, E_\alpha)(E - E_\alpha)$, namely, the form

$$\Delta(E) = \prod_\alpha C(E, E_\alpha)(E - E_\alpha) \tag{3.36}$$

of the spectral determinant is used. The regularizing function $C(E, E_\alpha)$ is chosen to make the product converge. It will not be specified, but it is assumed to be real and non-zero when E is real [37]. The expressions (3.3) and (3.34) for the resolvent Wigner function can be written as fractions and compared. Taking a common denominator and comparing between the numerators yields

$$\Delta(E) + [E - \mathcal{H}(\boldsymbol{x})]\mathcal{N}\left(\boldsymbol{x}; \frac{1}{\hbar}\right) = [E - \mathcal{H}(\boldsymbol{x})]\sum_\alpha C(E, E_\alpha)\bar{W}_\alpha(\boldsymbol{x}, \hbar) \times \prod_{\beta\neq\alpha} C(E, E_\beta)(E - E_\beta)\,. \tag{3.37}$$

The functional equation (3.27), together with the invariance property of the eigenvalues E_α under $\hbar$ reversal, lead to the corresponding symmetry of the right-hand side of (3.37), and consequently,

$$\mathcal{N}\left(x;\frac{1}{\hbar}\right)=\mathcal{N}\left(x;-\frac{1}{\hbar}\right). \tag{3.38}$$

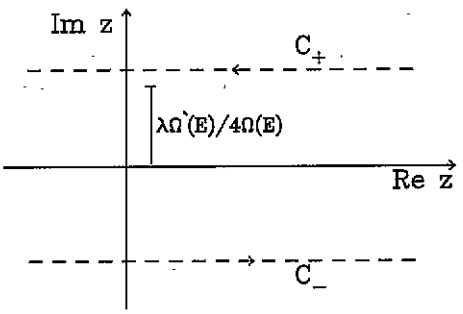

Figure 1. The integration contours C_+ and C_- in the z plane, and their relation to the entropy barrier (2.11) for $D=2$.

Note that the resulting numerator (3.37) of the resolvent Wigner function (3.3) is a continuous function of the energy E, therefore ε may be taken to be identically zero in (3.37) and (3.38).

By Cauchy's theorem $\mathcal{N}(x;1/\hbar)$ is expressed as a contour integral of the form (following [8])

$$\mathcal{N}\left(x;\frac{1}{\hbar}\right)=\frac{1}{2\pi\mathrm{i}}\int_{C_+ +C_-}\frac{\mathrm{d}z}{z}\gamma(z,\hbar)\mathcal{N}\left(x;\frac{1}{\hbar}+z\right) \tag{3.39}$$

where $C_\pm$ are the contours shown in figure 1. The function $\gamma(z,\hbar)$ is even in z, analytic within the integration strip, and satisfies $\gamma(0,\hbar)=1$. This is an analytic continuation of $\mathcal{N}$ from the region of complex $1/\hbar$ where the sum (3.35) is absolutely convergent to the real $1/\hbar$ axis. It assumes the analyticity of $\mathcal{N}$ in a sufficiently wide strip around the real axis. A similar assumption about the spectral determinant was made in [8]. With the aid of relation (3.38) one obtains

$$\mathcal{N}\left(x;\frac{1}{\hbar}\right)=\frac{1}{2\pi\mathrm{i}}\int_{C_+}\frac{\mathrm{d}z}{z}\gamma(z,\hbar)\left[\mathcal{N}\left(x;z+\frac{1}{\hbar}\right)+\mathcal{N}\left(x;z-\frac{1}{\hbar}\right)\right]. \tag{3.40}$$

Choosing the integartion path C_+ sufficiently far from the real axis of $1/\hbar$ so that condition (2.11) holds ensures that the triple sum of (3.35) converges everywhere on C_+. Therefore one may substitute (3.35) into (3.40) to obtain

$$\mathcal{N}\left(x;\frac{1}{\hbar}\right)=4\pi\sum_{p,n,\mu}\left[U_{p,n,\mu}\left(\frac{1}{\hbar}\right)+U_{p,n,\mu}\left(-\frac{1}{\hbar}\right)\right] \tag{3.41}$$

where

$$U_{p,n,\mu}\left(\frac{1}{\hbar}\right)=-\mathrm{i}c_\mu^{(p,n)}g_p^{(n)}(\partial_\eta)\mathrm{e}^{-\mathrm{i}\pi\tilde{N}[E;(1/\hbar)]+(\mathrm{i}/\hbar)\tilde{S}_\mu^p(x,\eta)}\frac{1}{2\pi\mathrm{i}}\int_{C_+}\frac{\mathrm{d}z}{z}\gamma(z,\hbar)A^p$$
$$\times\left(x,E;\frac{1}{\hbar}+z\right)\mathrm{e}^{\mathrm{i}\{\pi\tilde{N}[E;(1/\hbar)]-\pi\tilde{N}[E;(1/\hbar)+z]+z\tilde{S}_\mu^p(x,\eta)\}}. \tag{3.42}$$

Note the slight change in the definition of the arguments of the Airy factor $A^p(x, E; 1/\hbar)$ and the mean level staircase $\bar{N}(E; 1/\hbar)$. It is introduced in order to demonstrate the explicit dependence of these terms on $1/\hbar$.

A similar relation to (2.13), namely

$$c_\mu^{(p,n)} e^{-i\pi\bar{N}(E)+(i/\hbar)\bar{\mathcal{S}}_\mu^p(x,\eta)} \to \left[c_\mu^{(p,n)} e^{-i\pi\bar{N}(E)+(i/\hbar)\bar{\mathcal{S}}_\mu^p(x,\eta)}\right]^* \qquad \text{if} \quad \hbar \to -\hbar \tag{3.43}$$

holds also for the pseudo orbits of the resolvent Wigner function. This relation follows from the formula (C.11) for $c_\mu^{(p,n)}$, since under $\hbar$ reversal the Maslov phases of that expression and the mean level staircase $\bar{N}(E)$ change their signs. One may understand these sign changes by examination of the time-integral representation of the resolvent operator (3.1) which under $\hbar$ reversal transforms into

$$\hat{\mathcal{R}} = \frac{i}{\hbar}\int_0^\infty dt\, e^{-(i/\hbar)(E-\hat{\mathcal{H}}^*)t+\varepsilon t/\hbar}\,. \tag{3.44}$$

Note that the convergence of the above integral is achieved for negative values of ε. Thus under $\hbar$ reversal $\varepsilon \to -\varepsilon$, and therefore the imaginary part of the resolvent operator also changes its sign. The density of states is proportional to the trace of this object, therefore, also the mean energy staircase $\bar{N}(E)$ changes its sign when $\hbar \to -\hbar$. The Maslov phases change sign under $\hbar$ reversal since the Wigner propagator $K_W(x; t)$ of (3.4) is transformed into its complex conjugate under $\hbar$ reversal. Note that also the functions $g_p^{(n)}(\partial_\eta)$ in the triple sum, which do not depend explicitly on $\hbar$, are invariant under $\hbar$ reversal, while the Airy factor (3.8) changes sign under $\hbar \to -\hbar$.

From these properties of the terms which construct $U_{p,n,\mu}(1/\hbar)$, and by an argument which involves deformation of the integration path C_+ to the real axis, one concludes that

$$U_{p,n,\mu}\left(-\frac{1}{\hbar}\right) = U^*_{p,n,\mu}\left(\frac{1}{\hbar}\right)\,. \tag{3.45}$$

Therefore using (3.41) one may express the triple sum of (3.35) as a manifestly real function, namely

$$\mathcal{N}\left(x; \frac{1}{\hbar}\right) = 8\pi\ \mathrm{Re}\sum_{p,n,\mu} U_{p,n,\mu}\left(\frac{1}{\hbar}\right)\,. \tag{3.46}$$

This formula makes sense only if all the sums and integrals converge. This may be achieved by choosing [8]

$$\gamma(z, \hbar) = e^{-\frac{1}{2}K^2 z^2|\hbar|} \tag{3.47}$$

where K is a constant which plays a similar role to the one introduced for the spectral determinant (see section 2).

In order to perform the integral (3.42), the Airy factor is represented as an integral of the form

$$A^p\left(x, E; \frac{1}{\hbar}+z\right) = \frac{1}{2\pi}\left(\frac{1}{\hbar}+z\right)\int d\tau\, e^{i[(1/\hbar)+z]\varphi_p(\tau)} \tag{3.48}$$

where

$$\varphi_p(\tau) = \frac{\tau^3}{24}\ddot{x}\wedge\dot{x} + [\mathcal{H}(x) - E]\tau\,. \tag{3.49}$$

The exponent factor is expanded as

$$e^{i\{\pi\bar{N}[E;(1/\hbar)]-\pi\bar{N}[E;(1/\hbar)+z]\}} = e^{-i\pi[\bar{N}_1 z+\frac{1}{2}\bar{N}_2 z^2]}\left[1+\sum_{m=3}^{\infty} z^m v_m\right] \tag{3.50}$$

where

$$\begin{aligned}\bar{N}_1 &\approx \frac{\Omega(E)}{h^2}\\ \bar{N}_1 &= \frac{\partial}{\partial(1/\hbar)}\bar{N}\left(E;\frac{1}{\hbar}\right) \approx \frac{\Omega(E)}{\pi h}\\ \bar{N}_2 &= \frac{\partial^2}{\partial(1/\hbar)^2}\bar{N}\left(E;\frac{1}{\hbar}\right) \approx \frac{\Omega(E)}{2\pi^2}\end{aligned} \tag{3.51}$$

and v_m are the coefficients of the expansions. The asymptotic expansion of the triple sum is now obtained by integration over z. The higher terms of the expansion (in the terminology of [8]) correspond to the terms of higher power in z. Here we will confine our attention only to the lowest-order term of the m expansion corresponding to (3.50). This is given by

$$U_{p,n,\mu}\left(\frac{1}{\hbar}\right) \approx \frac{-i}{2h}\int d\tau\, c_\mu^{(p,n)} g_p^{(n)}(\partial_\eta) e^{-i\pi\bar{N}[E;(1/\hbar)]+(i/\hbar)[\bar{\mathcal{S}}_\mu^p(x,\eta)+\varphi_p(\tau)]}(J_{\mu,p}+I_{\mu,p}) \tag{3.52}$$

where

$$J_{\mu,p} = \frac{1}{\pi i}\int_{C_+}\frac{dz}{z}\gamma(z,\hbar)\, e^{-i\pi[\bar{N}_1 z+\frac{1}{2}\bar{N}_2 z^2]+iz[\bar{\mathcal{S}}_\mu^p(x,\eta)+\varphi_p(\tau)]} \tag{3.53}$$

while

$$I_{\mu,p} = \frac{\hbar}{\pi i}\int_{C_+} dz\,\gamma(z,\hbar) e^{-i\pi[\bar{N}_1 z+\frac{1}{2}\bar{N}_2 z^2]+iz[\bar{\mathcal{S}}_\mu^p(x,\eta)+\varphi_p(\tau)]}. \tag{3.54}$$

Performing the integration over z yields

$$J_{\mu,p} = \mathrm{Erfc}\left\{\frac{\bar{\xi}_p(\mu,x,E,\eta)+\varphi_p(\tau)}{B(K,\hbar,E)\sqrt{2\hbar}}\right\} \tag{3.55}$$

and

$$I_{\mu,p} = i\frac{\sqrt{\hbar}}{B(K,\hbar,E)}\exp\left\{-\frac{[\bar{\xi}_p(\mu,x,E,\eta)+\varphi_p(\tau)]^2}{2\hbar B^2(K,\hbar,E)}\right\} \tag{3.56}$$

where $B(K,\hbar,E)$ is given by (2.17) for $D=2$, while K is a free-tuning parameter, and

$$\bar{\xi}_p(\mu,x,E,\eta) \approx \bar{\mathcal{S}}_\mu^p(x,\eta) - \frac{\Omega(E)}{2\pi\hbar}. \tag{3.57}$$

Finally one finds

$$\begin{aligned}\mathcal{N}\left(x;\frac{1}{\hbar}\right) = -\mathrm{Re}\frac{4\pi i}{h}\int d\tau \sum_{p,n,\mu} c_\mu^{(pn)} g_p^{(n)}(\partial_\eta)\\ \times e^{-i\pi\bar{N}(E)+(i/\hbar)[\bar{\mathcal{S}}_\mu^p(x,\eta)+\varphi_p(\tau)]}(J_{\mu,p}+I_{\mu,p}).\end{aligned} \tag{3.58}$$

Formula (3.58) is the leading term in the asymptotic m series resulting from (3.48) and (3.50). Taking the derivatives with respect to η results in two sorts of contributions. One comes from differentiation of the exponents $\exp\{-i\pi\bar{N}(E_\alpha)+(i/\hbar)\mathcal{S}_\mu^p(x,\eta)\}$, while the others result from differentiation of the complementary error functions $J_{\mu,p}$, or the Gaussians $I_{\mu,p}$. Because the contributions of the second sort are proportional to Gaussians centred at (2.14), these will pick up their contributions only from pseudo-orbits in the vicinity of the smoothed cut-off (2.19). However, from the structure of the asymptotic expansion described in section 2 for the spectral determinant, it is clear that these contributions correspond to higher orders of the full asymptotic expansion. Therefore, confining our attention only to the main sum of the expansion, the result is obtained by considering only terms which after the differentiation are multiplied by the complementary error function term. These may be obtained by substitution of (3.20) and (3.21) for $g_p^{(n)}(\partial_\eta)$ and setting $\eta = 1$. Thus, the approximate semiclassical formula for the resolvent Wigner function is

$$W(x,E) \approx W_0(x,E) + \frac{4\pi}{\Delta(E)}\mathrm{Re}\left\{-i\sum_{p,n,\mu}\int\frac{d\tau}{h}c_\mu^{(p,n)}g_p^{(n)}(b)\right.$$
$$\left.\times\, e^{-i\pi\bar{N}(E)+(i/\hbar)[\mathcal{S}_\mu^p(x)+\varphi_p(\tau)]}\mathrm{Erfc}\left\{\frac{\xi_p(\mu,x,E)+\varphi_p(\tau)}{B(K,\hbar,E)\sqrt{2\hbar}}\right\}\right\} \tag{3.59}$$

where

$$\mathcal{S}_\mu^p(x) = \bar{\mathcal{S}}_\mu^p(x,\eta=1) = \mathcal{S}_\mu + S_p + \bar{X}R_pX \tag{3.60}$$

and

$$\xi_p(\mu,x,E) = \bar{\xi}_p(\mu,x,E,\eta=1) \approx \mathcal{S}_\mu^p(x) - \frac{\Omega(E)}{2\pi\hbar}. \tag{3.61}$$

The formula for the Wigner–Weyl representation of the eigenstate ψ_α, which is the central result of this paper, may now be obtained from (3.59). In the view of level repulsion in chaotic systems, one may assume that almost all the zeros of $\Delta(E)$ are simple. Thus if E_α is an eigenvalue then when $E \to E_\alpha$ the spectral determinant takes the form $\Delta(E) = \Delta'(E_\alpha)(E - E_\alpha)$, and in the limit $\varepsilon \to 0$, one may identify the corresponding Wigner function $W_\alpha(x)$ of (3.3) as the residue of the expression (3.59) at $E = E_\alpha$ divided by h^2, thus

$$W_\alpha(x) \approx \frac{4\pi}{h^2\Delta'(E_\alpha)}\mathrm{Re}\left\{-i\sum_{p,n,\mu}\int\frac{d\tau}{h}c_\mu^{(p,n)}g_p^{(n)}(b)\right.$$
$$\left.\times\, e^{-i\pi\bar{N}(E_\alpha)+(i/\hbar)[\mathcal{S}_\mu^p(x)+\varphi_p(\tau)]}\mathrm{Erfc}\left\{\frac{\xi_p(\mu,x,E_\alpha)+\varphi_p(\tau)}{B(K,\hbar,E_\alpha)\sqrt{2\hbar}}\right\}\right\}. \tag{3.62}$$

Note that the contribution from $W_0(x,E)$ which comes from the zero-length orbits takes the form

$$W_\alpha^0(x) = \begin{cases} 1/h^2 & \text{if } \mathcal{H}(x) = E_\alpha \\ 0 & \text{otherwise.} \end{cases} \tag{3.63}$$

However, it is defined only on a subset of measure zero of the phase space, thus for any practical application, such as calculation of matrix elements etc, its contribution vanishes. This term is therefore dropped from (3.62).

The phase $\varphi_p(\tau)$ is for most of the pseudo-orbits negligible compared with $\xi_p(\mu, x, E_\alpha)$. The reason is that $\varphi_p(\tau)$ is a local term corresponding to properties of the close vicinity of the periodic orbit p. Note that the derivation of (3.7) and (3.8) in [30] is based on linearization in this vicinity, therefore $\varphi_p(\tau)$ is much smaller than the corresponding action of the periodic orbit p. Thus for almost all the pseudo-orbits the corresponding complementary error function term is practically independent of $\varphi(\tau)$. For these the integration over τ may be performed recovering the Airy term (3.8). This argument does not hold for the pseudo-orbits with periods near the cut-off, resulting from the error function. However, these contribute to higher orders of the asymptotic expansion. Thus, confining our attention only to the lowest order of the asymptotic expansion reduces (3.62) to

$$W_\alpha(x) \approx \frac{4\pi}{\hbar^2 \Delta'(E_\alpha)} \mathrm{Re}\left\{ -\mathrm{i} \sum_{p,n,\mu} A^p(x, E_\alpha) c_\mu^{(p,n)} g_p^{(n)}(b) \right.$$
$$\left. \times \mathrm{e}^{-\mathrm{i}\pi \tilde{N}(E_\alpha) + (\mathrm{i}/\hbar)\mathcal{S}_\mu^p(x)} \mathrm{Erfc}\left\{ \frac{\xi_p(\mu, x, E_\alpha)}{B(K, \hbar, E_\alpha)\sqrt{2\hbar}} \right\} \right\}. \tag{3.64}$$

The above formula, which represents the Wigner function corresponding to the eigenstate ψ_α, as a sum over periodic orbits, is the central result of this paper. In principle it enables us to calculate the eigenstates in the semiclassical approximation. Due to rapid oscillations of the contributions from orbits that are distant from x, where approximation (3.7) does not hold, these are expected to average out to zero.

It is simple to show that a similar derivation to that of the spectral determinant yields a formula for $\Delta'(E)$ appearing in (3.64). To the lowest order of the asymptotic expansion it is given by

$$\Delta'(E) \approx \mathrm{Im} \sum_\mu \left(\pi \bar{\mathrm{d}}(E) - \frac{\mathcal{T}_\mu}{\hbar} \right) c_\mu \mathrm{e}^{-\mathrm{i}\pi \tilde{N}(E) + (\mathrm{i}/\hbar)\mathcal{S}_\mu} \mathrm{Erfc}\left\{ \frac{\xi(\mu, \hbar, E)}{B(K, \hbar, E)\sqrt{2\hbar}} \right\} \tag{3.65}$$

as can be seen from differentiation of (2.15). All the quantities involved in this formula are defined in section 2.

4. The probability density

One may obtain the probability density, $\rho_\alpha(q) = |\psi_\alpha(q)|^2$, simply by projecting (3.64) for the Wigner function onto the configuration space. Similar to the case of the Wigner function, the following discussion will be restricted to the lowest order of the asymptotic expansion.

The integration over the momenta will be performed using the canonical variables defined by (3.9). Integrating over $\mathcal{H}$ is straightforward using the normalization property of the Airy function,

$$\int \mathrm{Ai}(u)\, \mathrm{d}u = 1\,. \tag{4.1}$$

The integration over the momenta P of the Poincaré surface of section, for each one of the primitive periodic orbits, involves the quadratic term,

$$\tilde{X} R_p X = R_{22}^p \left(P + \frac{R_{12}^p Q}{R_{22}^p} \right)^2 - \frac{Q^2}{R_{22}^p} \tag{4.2}$$

where R^p_{12} and R^p_{22} are the matrix elements of $\boldsymbol{R}_p$ defined by (3.14). The integration is complicated since this term appears also in the complementary error function $\mathrm{Erfc}\{\xi_p(\mu, x, E_\alpha)/B(K,\hbar,E_\alpha)\sqrt{2\hbar}\}$. Yet, the complementary error function term may be expanded arround $\boldsymbol{X} = (Q, P = -R^p_{12}Q/R^p_{22})$. This expansion consists of this function evaluated at this point, and higher-order terms which are proportional to Gaussians centred at (2.14). However, the higher-order terms correspond to higher orders of the asymptotic expansion and for a calculation in the lowest order these may be ignored. It is now convenient to express the functions $g^{(n)}_p$ in terms of the derivatives ∂_η as described in section 3. Thus, to the lowest order, the integration over P becomes a trivial Gaussian integral, and after changing the variable t (which is the time along the periodic orbit measured from the Poincaré section) to q, which is the coordinate along the orbit, one obtains

$$\rho_\alpha(q) = \frac{4\pi}{h^2\Delta'(E_\alpha)}\mathrm{Re}\left\{-\mathrm{i}\sum_{p,n,\mu} c^{(pn)}_\mu g^{(n)}_p(\partial_\eta)\sqrt{\frac{\mathrm{i}\hbar\pi}{\eta R^p_{22}}}\right.$$
$$\left.\times\frac{1}{|\dot q|}\mathrm{e}^{-\mathrm{i}\pi\bar N(E_\alpha)+(\mathrm{i}/\hbar)S^{p,\mu}(q,\eta)}\mathrm{Erfc}\left\{\frac{\breve\xi_p(\mu,q,E_\alpha)}{B(K,\hbar,E_\alpha)\sqrt{2\hbar}}\right\}\right\}\Bigg|_{\eta=1} \tag{4.3}$$

where

$$S^{p,\mu}(q,\eta) = S_\mu + S_p - \eta\frac{Q^2}{R^p_{22}} \tag{4.4}$$

and

$$\breve\xi_p(\mu,q,E_\alpha) = S_\mu + S_p - \frac{Q^2}{R^p_{22}} - \frac{\Omega(E)}{2\pi\hbar} \tag{4.5}$$

while Q is the coordinate transverse to the orbit, and $\dot q$ is the velocity along the orbit.

The pattern of the quadratic fringes, in the vicinity of the periodic orbit p, changes along the orbit according to the factor $R^p_{22}(q)$. Singular points appear whenever $R^p_{22} = 0$. These points are the self-focal points, and a considerable enhancement of $\rho_\alpha(q)$ is associated with them [29]. However, similar to caustics, at the self-focal points and their vicinity, the semiclassical approximation (4.3) is not valid. A formula for these regions may be obtained as well.

Formula (4.3) for the probability density of individual eigenstates is the resummed version of Bogomolny's formula [29]. In the latter the convergence of the periodic orbit sum is achieved by averaging over a small interval of energy, therefore it is inadequate for describing individual wavefunctions.

5. Scars, and the $\Delta^{(p,n)}(E)$ functions

Some of the features concerning the formula for the semiclassical approximation of the Wigner function (3.64) will now be discussed. In particular, we will be interested in the mechanism which leads to a scarred wavefunction. It will be argued that this mechanism is related to the functions $\Delta^{(p,n)}(E)$ defined by (3.26).

The main character which formula (3.64) reveals is that only a finite number of primitive periodic orbits participate in the construction of the eigenfunction at a given energy E_α. For ergodic systems these are the orbits with period T_p satisfying approximately

$$T_p < \pi\hbar\bar{d}(E_\alpha) + K\sqrt{2\hbar}\frac{\Omega'(E_\alpha)}{2\Omega(E_\alpha)}. \tag{5.1}$$

Condition (5.1) follows from the factor of the complementary error function, where the influence of the local term $\tilde{\boldsymbol{X}}\boldsymbol{R}_p\boldsymbol{X}$ is ignored since it is negligible compared with the corresponding periodic orbit action S_p. It also reveals the existence of an 'analytical bootstrap' between long and short periodic orbits, suggested by Berry [30]. The set of primitive orbits contributing at a given energy is approximately the one that contributes to the spectral determinant. It is satisfying that this 'analytical bootstrap' extends beyond the spectral determinant and also corresponds to the structure of the wavefunctions in phase space.

A series of functions, labelled by the index n, is associated with each one of the primitive periodic orbits. From the last factor in the expression (3.26) for $\Delta^{(p,n)}(E)$, which is proportional to $\mathrm{e}^{-u_p(\frac{1}{2}+n)}$, it is obvious that usually the dominant contribution will come from the terms with $n = 0$. However if the instability exponent u_p is small for some particular orbit, one should consider also the higher terms due to $n = 1, 2, 3$, etc.

This zero term $n = 0$ forms a pattern of quadratic fringes in phase space transverse to the orbit with spacing between fringes of order $h^{\frac{1}{2}}$ near the periodic orbit, and of order h far from it. The fringes pattern changes along the orbit since the eigenvectors of the monodromy matrix change and, therefore, also the matrix $\boldsymbol{R}_p(\boldsymbol{x})$.

However, when u_p is small, contributions from higher values of n may also be important and these will interfere. The nth term consists of the quadratic fringes factor $\mathrm{e}^{-(\mathrm{i}/\hbar)\mathcal{S}_\mu^p(\boldsymbol{x})}$ multiplied by the function $g_p^{(n)}(b)$ which is a polynomial of order n in the variable $b = (\mathrm{i}/\hbar)\tilde{\boldsymbol{X}}\boldsymbol{R}_p\boldsymbol{X}$, therefore for small u_p, as n increaces, these corrections may become significant at regions which are more distant from the location of the primitive periodic orbit p.

We turn now to re-examine the expression (3.25) for the resolvent Wigner function at complex values of $1/\hbar$. For the corresponding Wigner functions (evaluated at complex $1/\hbar$), each one of the $g_p^{(n)}$ terms described above is multiplied by a different function of the energy, namely $\Delta^{(p,n)}(E)$. Unfortunately, after the analytic continuation these functions lose their individual meaning, since the functional equation (3.38) holds for the whole numerator $\mathcal{N}(\boldsymbol{x}; 1/\hbar)$ and not for each one of the $\Delta^{(p,n)}(E)$ functions.

When a wavefunction is scarred and only few periodic orbits contribute to (3.64), one expects these functions (or related functions) also to have a meaning for real values of $\hbar$. For this purpose it is instructive to integrate the Wigner function over a small domain in phase space Γ_p which surrounds the primitive periodic orbit p. The tube Γ_p surounding the orbit is assumed to be sufficiently large so that the integration may be performed by the stationary phase approximation. The integration becomes simple using the variables $\mathcal{H}$, t, and $\boldsymbol{X}(Q, P) = (Q_1, \ldots Q_{D-1}, P_1, \ldots P_{D-1})$ defined by (3.9). Integration over $\mathcal{H}$ is straightforward due to (4.1). Before integrating over $\boldsymbol{X}$ it will be convenient to express $g_p^{(n)}$ in terms of the derivatives ∂_η as described in section 3, and to expand the complementary error function term around $\boldsymbol{X} = 0$. Limiting ourselves only to the lowest order of the asymptotic expansion implies that the only contributing term is the zeroth-order one, where the complementary error function is evaluated at $\boldsymbol{X} = 0$. The higher terms of the expansion in $\boldsymbol{X}$ are proportional to Gaussians centred at the cut-off, therefore these

pick up their contribution from a small number of pseudo-orbits. These are the ones which satisfy (2.19). The integration over $\boldsymbol{X}$ is now simple, and noting that $\mathrm{Det}(\boldsymbol{R}_p) = -1$ leaves us with an integrand which is independent of t. Thus after the integration over t one obtains

$$\int_{\Gamma_p} W_\alpha(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \frac{T_p}{\hbar\Delta'(E_\alpha)}\mathrm{Re}\left\{-\mathrm{i}\sum_{n,\mu} c_\mu^{(p,n)} \mathrm{e}^{-\mathrm{i}\pi\bar{N}(E_\alpha)+(\mathrm{i}/\hbar)\mathcal{S}_{\mu,p}} \times g_p^{(n)}(\partial_\eta)\frac{1}{\eta}\mathrm{Erfc}\left\{\frac{\mathcal{S}_{\mu,p}-\Omega(E_\alpha)/h}{B(K,\hbar,E_\alpha)\sqrt{2\hbar}}\right\}\right\} \tag{5.2}$$

where T_p is the period of the primitive orbit p, $\mathcal{S}_{\mu,p}$ is the pseudo-orbit action defined in (3.30) and $\Omega(E_\alpha)$ is the classical phase-space volume with energy less than E_α. After taking the derivatives, η should be set to 1. It is simple to verify that $g_p^{(n)}(\partial_\eta)1/\eta|_{\eta=1} = 1$ for all n. This result suggests that it is plausible to define the $\bar{\Delta}^{(p,n)}(E)$ functions at real values of $\hbar$ to be

$$\bar{\Delta}^{(p,n)}(E) = \mathrm{Re}\left\{-\mathrm{i}\sum_{\mu} c_\mu^{(p,n)} \mathrm{e}^{-\mathrm{i}\pi\bar{N}(E_\alpha)+(\mathrm{i}/\hbar)\mathcal{S}_{\mu,p}}\mathrm{Erfc}\left\{\frac{\mathcal{S}_{\mu,p}-\Omega(E_\alpha)/h}{B(K,\hbar,E_\alpha)\sqrt{2\hbar}}\right\}\right\}. \tag{5.3}$$

These are the resummed versions of (3.29). For a state that is scarred by a periodic orbit p^*, only terms corresponding to this orbit are expected to be important in the sum (3.25). For such a state, $\Delta^{(p^*,n)}(E)$ can be approximated by $\bar{\Delta}^{(p^*,n)}(E)$, and no further resummation is required. For states that are not scarred by the p^*th orbit, $\Delta^{(p^*,n)}(E)$ have no meaning for real values of $\hbar$. In the view of the results (5.2) and (5.3) it is instructive to define the weight of a scar for the eigenstate ψ_α, which corresponds to the primitive periodic orbit p by

$$Y_p(E_\alpha) = \int_{\Gamma_p} W_\alpha(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \frac{T_p}{\hbar\Delta'(E_\alpha)}\Lambda_p(E_\alpha) \tag{5.4}$$

where

$$\Lambda_p(E) = \sum_{n=0}^{\infty} \bar{\Delta}^{(p,n)}(E)\,. \tag{5.5}$$

The normalization property of the Wigner function,

$$\int W_\alpha(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 1 \tag{5.6}$$

implies that the scar weight $Y_p(E_\alpha)$ cannot exceed 1 provided the tube Γ_{p^*} is sufficiently wide. Nevertheless, when an eigenstate ψ_α is scarred along some orbit p^*, one would expect its corresponding weight $Y_{p^*}(E_\alpha)$ to be close to one, while all the others approach zero. It turns out that the function $Y_p(E)$, expressed in terms of periodic orbits, provides a useful tool in order to predict the appearance of scars along the periodic orbit p. For this purpose one has to calculate this function on the spectrum, and to check at which eigenenergies it approaches 1. The corresponding wavefunctions will be scarred along the orbit p.

The functions $\Lambda_p(E)$ appeared recently in a different context in which the diagonal matrix elements of a smooth operator were calculated semiclassically [38]. Let $\mathcal{O}$ be a smooth operator. Its diagonal matrix element corresponding to the state ψ_α is semiclassically given by

$$\langle\psi_\alpha \mid \mathcal{O} \mid \psi_\alpha\rangle = \frac{-\mathrm{i}}{\hbar}\left(\frac{\mathrm{d}Z}{\mathrm{d}E}\right)^{-1}\frac{\partial Z}{\partial\eta}\bigg|_{\substack{\eta=0\\ E=E_\alpha}} \tag{5.7}$$

where

$$Z = \mathrm{e}^{-\mathrm{i}\pi\bar{N}(E)} \prod_{\mathrm{ppo}} \prod_{n} \left\{ 1 - \exp\left[\eta \mathcal{O}_p + \frac{\mathrm{i}}{\hbar} S_p(E) - \mathrm{i}\gamma_p - \left(\frac{1}{2} + n \right) u_p \right] \right\} \tag{5.8}$$

and

$$\mathcal{O}_p = \int_0^{T_p} \mathrm{d}t\, \mathcal{O}(q_p(t), p_p(t)) \,. \tag{5.9}$$

Up to the factor $\mathrm{e}^{-\mathrm{i}\pi\bar{N}(E)}$ which is inserted here for convenience, Z is the zeta function for the matrix elements. Its structure is similar to that of the dynamical zeta function (2.1) except it contains an additional term $\mathcal{O}_p$ which is the classical value of the operator $\mathcal{O}$ integrated along the primitive periodic orbit p. Taking the derivative with respect to η in (5.7) and substituting $\eta = 0$ results in

$$\langle \psi_\alpha \mid \mathcal{O} \mid \psi_\alpha \rangle = \frac{1}{\hbar \Delta'(E_\alpha)} \sum_p \mathcal{O}_p \Lambda_p(E_\alpha) \tag{5.10}$$

as is clear from (3.29) and (3.26). One may choose, now, the operator $\mathcal{O}$ to be the characteristic function over small region Γ_p along some particular periodic orbit p in the phase space. Clearly this choice will reduce (5.10) to (5.4). Ignoring convergence problems, it is clear that the expression (5.7) for the diagonal matrix elements is an approximation which one may obtain from the general semiclassical formula for the diagonal matrix elements using the Wigner functions

$$\langle \psi_\alpha \mid \mathcal{O} \mid \psi_\alpha \rangle = \int \mathrm{d}x\, W_\alpha(x) \mathcal{O}(x) \,. \tag{5.11}$$

The semiclassical formula for the Wigner functions (3.64) also provides a way to calculate semiclassically off-diagonal matrix elements in terms of periodic orbits. However, this issue will be discussed elsewhere.

An important property satisfied by the functions $\Lambda_p(E)$ is their direct relation to the derivative of the spectral determinant (3.65) evaluated on the spectrum. It is given by the following sum rule:

$$\hbar \Delta'(E_\alpha) = \sum_p T_p \Lambda_p(E_\alpha) \,. \tag{5.12}$$

This sum rule is a manifestation of the normalization of the Wigner function (5.6). It can be obtained from (5.10) with the choice $\mathcal{O} = \hat{1}$. One may derive it directly by differentiation, with respect to the energy, of the spectral determinant, represented as an Euler product (2.3), and analytic continuation of the result to the real $1/\hbar$ axis.

The understanding of the functions $\Lambda_p(E)$ or $\bar{\Delta}^{(p,n)}(E)$ is therefore of great importance, and it is crucial in order to provide a coherent explanation to the appearance of scars in individual wavefunctions. Obviously these will appear whenever one particular factor $\bar{\Delta}^{(p^*,0)}(E)$ (or few in more general cases) is significant while all the other are negligible. Note that when $\bar{\Delta}^{(p,n)}(E)$ is large, usually all $\bar{\Delta}^{(p,m)}(E)$ with $m < n$ are large too. Moreover, when the instability exponent is not too small, $\Lambda_p(E) \approx \bar{\Delta}^{(p,0)}(E)$. Therefore, usually, a study of the functions $\bar{\Delta}^{(p,0)}(E)$ will be sufficient for a crude estimate of the scar weights.

The exponential growth in the number of periodic orbits, as their action increases implies that scarred wavefunctions are likely to appear only at low enough energies

where the number of primitive periodic trajectories participating in the construction of the wavefunction is still small. At high energies, an exponentially large number of functions $\bar{\Delta}^{(p,n)}(E)$ contribute to the eigenstate, and the probability for a situation where only one or few of them are significant, while all the others negligible, becomes very small.

Furthermore, in the limit of high energy, significant contributions to the wavefunctions will come from long orbits as is clear from (3.64). For chaotic systems these explore almost the whole energy surface uniformly [35]. However, the scars do not get more concentrated as $T_p \to \infty$. The fringe spacing, that is of order $h^{\frac{1}{2}}$ near the periodic orbit and of order h far from the trajectory, is independent of the instability exponent. Therefore contributions of long periodic orbits will overlap, and the pattern of scars will probably be washed out. It is still not clear how to calculate these overlaps. As speculated by Berry [31], it is reasonable that the collective superposition of the long orbits results in a Gaussian random function decorating a smooth background.

These arguments suggest that scarred wavefunctions become increasingly rare as $E \to \infty$. We argue that scars will also not appear in the eigenstates of the low-energy limit. The reason is that a scarred wavefunction describes a particle which moves in a relatively narrow region in the configuration space, therefore its momentum is high and hence also its kinetic energy. Thus, only for a restricted range of energy scars are expected to appear. In this sense, scarred wavefunctions are rare in chaotic systems.

There is still a lack of understanding regarding the functions $\bar{\Delta}^{(p,n)}(E)$. Yet, there are some indications that at least in the low-energy regime, the spectral determinant (2.3) may be approximated by the real part of the truncated product [34]. Investigating the possibility for a similar behaviour for $\bar{\Delta}^{(p,n)}(E)$ will provide further theoretical understanding of the appearance of scars.

Suppose that at a certain eigenenergy E_α there is a situation where one of the terms in the truncated product of the dynamical zeta function (2.1),

$$F_{p^*} = \left(1 - e^{(i/\hbar)S_{p^*}(E_\alpha) - i\gamma_{p^*} - \frac{1}{2}u_{p^*}}\right) \tag{5.13}$$

is very small compared with all the others. Such a situation is possible when the instability exponent u_{p^*} is close to zero, that is when p^* corresponds to a short periodic orbit. Except in the function $\Delta^{(p^*,0)}(E_\alpha)$, this factor appears in all $\Delta^{(p,n)}(E_\alpha)$ where $p \neq p^*$. Assuming that the spectral determinant as well as the functions $\Delta^{(p,n)}(E_\alpha)$ can be approximated for some purposes by the truncated product, one concludes that $\Delta^{(p,n)}(E_\alpha)$ are proportional to (5.13), and therefore very small. However, since this term is missing from the product of $\Delta^{(p^*,0)}(E_\alpha)$, this function is large compared to the others, therefore $\Lambda_{p^*}(E_\alpha)$ is dominant among all $\Lambda_p(E_\alpha)$ and the main contribution to the Wigner function $W_\alpha(x)$ will come from the primitive periodic orbits p^*. In agreement with previous results [18, 29, 30, 39], these arguments show that, usually, scarred wavefunctions will correspond to the short periodic orbits of the classical dynamics, for which the instability exponents are relatively small.

It is instructive to derive an expression for the scar weight $Y_{p^*}(E_\alpha)$ in which the dependence on the small instability constant u_{p^*} is clear. Inserting the sum rule (5.12) into (5.4) yields

$$Y_{p^*}(E_\alpha) = \frac{1}{1+C} \tag{5.14}$$

where

$$C = \sum_{p \neq p^*} \frac{T_p}{T_{p^*}} \frac{\Lambda_p(E_\alpha)}{\Lambda_{p^*}(E_\alpha)}. \tag{5.15}$$

The sum over p converges, since $\Lambda_p(E_\alpha) = 0$ for long periodic orbits that do not satisfy the bound (5.1). For such orbits $\Lambda_p(E)$ practically vanishes due to the Erfc term in (5.3) and since the corresponding pseudo-orbits periods $\mathcal{T}_{\mu,p} = \partial \mathcal{S}_{\mu,p}/\partial E$ are larger than T_p. Moreover, due to the above assumptions, each one of the $\Lambda_p(E_\alpha)$ with $p \neq p^*$ is proportional to the factor (5.13) where its smallest (absolute) value is $F_{p^*} \approx u_{p^*}/2$ for small values of u_{p^*}. Therefore,

$$C = \bar{C} u_{p^*} \tag{5.16}$$

where $\bar{C}$ is a constant factor independent of u_{p^*} (to the lowest order).

In more complicated cases, a scarred wavefunction may contain contributions from more than one periodic orbit. Such a situation was related to the avoided crossing of the energy levels [24, 26]. However, the structure of formula (3.64) for the Wigner function suggests that, in the general situation, all the primitive periodic orbits satisfying (5.1), contribute to the wavefunction with weights determined by the $\bar{\Delta}^{(p,n)}(E_\alpha)$ of (5.3).

6. Discussion

The main results of the paper are (3.62) and (3.64) for the resummed semiclassical Wigner function corresponding to an eigenstate. This enables us in principle to calculate eigenstates of chaotic systems in terms of classical periodic orbits, within the accuracy of the semiclassical approximation. The required number of periodic orbits is similar to the one required in order to obtain the energy spectrum. The reason is that $\tilde{\boldsymbol{X}} \boldsymbol{R}_p \boldsymbol{X}$ in (3.60) is bounded in phase space, while $\mathcal{S}_\mu$ and S_p grow with the length of the orbits. Therefore the effective truncation of the sums in (3.64) by the complementary error function is approximately the same as was found for the spectral determinant by Berry and Keating [8].

This bound on the required number of periodic orbits is similar to the bound obtained by Berry and Keating [8], Bogomolny [13], and Doron and Smilansky [15]. Since the number of periodic orbits proliferates exponentially with their length, in practice, (3.64) enables calculation of eigenfunctions of relatively low energies which are sufficiently high for the semiclassical approximation to apply. This is also the case for the semiclassical approximation for the spectrum, therefore it can be regarded as its extension from the spectrum to eigenfunctions.

Integration over the momenta in (3.64) leads to the probability density corresponding to an eigenstate. The resulting formula (4.3) is the resummed version of the corresponding quantity that was calculated by Bogomolny [29].

Equation (3.25) can be written as a logarithmic derivative of a zeta function following the derivation of (5.7) that was presented in [38]

Formula (5.4) for the effective scar strength is obtained by averaging of (3.64) over a small region in phase space. It is a resummed version of an equation that was obtained earlier by a somewhat different method by Eckhardt *et al* [38]. It enables us to calculate the scar strength for sufficiently low energies. This should enable us to predict scarring if the periodic orbits are known.

The main problems that one encounters in the calculation of eigenstates are similar to those encountered in the calculation of the eigenenergies. One would like to find a method that does not require an exponentially large number of periodic orbits, similar to the method that was proposed for the calculation of the spectrum [16].

The analyticity of $\mathcal{N}(x; (1/\hbar) + z)$ that was assumed in (3.39) for sufficiently wide strip around the real z axis requires rigorous treatment that was not attempted in the present work.

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Appendix A. Expansion of the term $\tilde{X}J[(M_p - I)/(M_p + I)]X$

In this appendix formula (3.13) is proved. Starting from expression (3.12) for the monodromy matrix M_p, one finds

$$J\frac{M_p - I}{M_p + I} = J\frac{1}{v_p^+ \wedge v_p^-}\begin{pmatrix} v_{p1}^+ & v_{p1}^- \\ v_{p2}^+ & v_{p2}^- \end{pmatrix} \times \begin{pmatrix} (e^{u_p} - 1)/(e^{u_p} + 1) & 0 \\ 0 & (e^{-u_p} - 1)/(e^{-u_p} + 1) \end{pmatrix}\begin{pmatrix} v_{p2}^- & -v_{p1}^- \\ -v_{p2}^+ & v_{p1}^+ \end{pmatrix} \tag{A.1}$$

where J is the unit symplectic matrix given by (3.10) and $v_p^\pm$ are the eigenvectors of the monodromy matrix; thus,

$$M_p v_p^\pm = e^{\pm u_p} v_p^\pm . \tag{A.2}$$

Extraction of the factor $(1 - e^{-u_p})/(1 + e^{-u_p})$ from (A.1) yields

$$J\frac{M_p - I}{M_p + I} = \frac{1 - z}{1 + z}\frac{1}{v_p^+ \wedge v_p^-}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} v_{p1}^+ & v_{p1}^- \\ v_{p2}^+ & v_{p2}^- \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} v_{p2}^- & -v_{p1}^- \\ -v_{p2}^+ & v_{p1}^+ \end{pmatrix}$$

where $z = e^{-u_p}$. After multiplying the matrices one obtains

$$J\frac{M_p - I}{M_p + I} = \frac{1 - z}{1 + z}\frac{-1}{v_p^+ \wedge v_p^-}\begin{pmatrix} -2v_{p2}^+ v_{p2}^- & v_{p2}^+ v_{p1}^- + v_{p2}^- v_{p1}^+ \\ v_{p2}^+ v_{p1}^- + v_{p2}^- v_{p1}^+ & -2v_{p1}^+ v_{p1}^- \end{pmatrix} . \tag{A.3}$$

Finally, using

$$v_p^+ \wedge v_p^- = \tilde{v}_p^+ J v_p^- = v_{p1}^+ v_{p2}^- - v_{p2}^+ v_{p1}^- \tag{A.4}$$

and defining $R_p^\pm$ according to (3.15) leads to the required result (3.13).

Appendix B. The functional equation

In this appendix, the functional equation (3.27) for the Wigner function multiplied by $\hbar^D$ is proved. Let $\mathcal{H}$ be the Hamiltonian of a system satisfying the time-independent Schrödinger equation,

$$\mathcal{H} \mid \psi_\alpha\rangle = E_\alpha \mid \psi_\alpha\rangle . \tag{B.1}$$

When $\hbar \to -\hbar$ the Hamiltonian transforms to its conjugate $\mathcal{H}^*$, therefore its new eigenfunctions are $|\psi_\alpha^*\rangle$, namely,

$$\mathcal{H}^* |\psi_\alpha^*\rangle = E_\alpha |\psi_\alpha^*\rangle . \tag{B.2}$$

According to the definition of the Wigner function,

$$\bar{W}_\alpha(x,\hbar) = \int \mathrm{d}q' \, \mathrm{e}^{-(\mathrm{i}/\hbar)pq'} \psi_\alpha\left(q + \tfrac{1}{2}q'\right)\psi_\alpha^*\left(q - \tfrac{1}{2}q'\right) \tag{B.3}$$

and the corresponding expression for the $\hbar$-reversed Hamiltonian is

$$\bar{W}_\alpha(x,\hbar) = \int \mathrm{d}q' \, \mathrm{e}^{+(\mathrm{i}/\hbar)pq'} \psi_\alpha^*\left(q + \tfrac{1}{2}q'\right)\psi_\alpha\left(q - \tfrac{1}{2}q'\right) . \tag{B.4}$$

Manifestly, expressions (B.3) and (B.4) are complex conjugates of one another, and, since the Wigner function is a real function for real values of x, these two expressions are equal, leading to (3.27).

Appendix C. Dirichlet representation of $\Delta^{(p,n)}(E)$

In this appendix, the amplitudes $c_\mu^{(pn)}$ that are required for the Dirichlet sum $\Delta^{(p,n)}(E)$ of (3.29) are derived. First, in order to expand (3.26) one needs an extension of the Euler identity (2.5) for the case

$$\prod_{\substack{j=0 \\ j\neq n}}^{\infty}(1 - ax^j) = 1 - a\sum_{\substack{j_1=0 \\ j_1\neq n}}^{\infty} x^{j_1} + a^2 \sum_{\substack{j_1>j_2 \\ j_1,j_2\neq n}} x^{j_1+j_2} - \cdots . \tag{C.1}$$

For this purpose we define the functions

$$d_m(x) = \sum_{j_1>j_2>\cdots>j_m} x^{j_1+j_2+\cdots+j_m} = \frac{x^{m(m-3)/4}}{(x^{-\frac{1}{2}} - x^{\frac{1}{2}})(x^{-1} - x^1)\cdots(x^{-m/2} - x^{m/2})} \tag{C.2}$$

with $d_0(x) = 1$. Using these definitions, the Euler identity (2.5) may be written as

$$\prod_{j=0}^{\infty}(1 - ax^j) = \sum_{m=0}^{\infty}(-a)^m d_m(x) . \tag{C.3}$$

It will be convenient to introduce new functions

$$d_m^{(n)}(x) = \sum_{\substack{j_1>j_2>\cdots>j_m \\ j_1,j_2,\cdots j_m\neq n}} x^{j_1+j_2+\cdots+j_m} . \tag{C.4}$$

Thus the expression for (C.1) is obtained from (C.3) simply by replacing $d_m(x)$ by $d_m^{(n)}(x)$, namely

$$\prod_{\substack{j=0 \\ j\neq n}}^{\infty}(1 - ax^j) = \sum_{m=0}^{\infty}(-a)^m d_m^{(n)}(x) . \tag{C.5}$$

To obtain a simple expression for the functions $d_m^{(n)}(x)$, note that one may express (C.4) as

$$d_m^{(n)}(x) = \sum_{j_1>j_2>\cdots>j_m} x^{j_1+j_2+\cdots+j_m} - \sum_{l=1}^{m} \sum_{\substack{j_1>j_2>\cdots>j_m \\ j_l=n}} x^{j_1+j_2+\cdots+j_m} . \tag{C.6}$$

The first term on the right-hand side is clearly $d_m(x)$, while the second term is

$$-x^n \sum_{\substack{j_2>j_3>\cdots>j_m \\ j_2,j_3,\cdots j_m\neq n}} x^{j_2+j_3+\cdots+j_m} . \tag{C.7}$$

This follows from the fact that choosing one of the j_l to be equal to n forces all the others to be different from n. Therefore the recursive formula

$$d_m^{(n)}(x) = d_m(x) - x^n d_{m-1}^{(n)}(x) \tag{C.8}$$

is satisfied, leading to

$$d_m^{(n)}(x) = \sum_{j=0}^{m} (-1)^j x^{jn} d_{m-j}(x) . \tag{C.9}$$

For most applications only the term $d_m^{(0)}$ is required. In this case the calculation of $d_m^{(0)}(x)$ is trivial since $\prod_{j=1}^{\infty}(1-ax^j)$ reduces to (C.3) with a replaced by ax, leading to the identification

$$d_m^{(0)}(x) = x^m d_m(x) . \tag{C.10}$$

Using this notation it is straightforward to see that the amplitudes $c_\mu^{(pn)}$ in (3.29) are given by

$$c_\mu^{(p,n)} = \prod_{p'\neq p} (-1)^{m_{p'}} \, \mathrm{e}^{-u_{p'}/2 m_{p'}} \, \mathrm{e}^{-\mathrm{i}\gamma_{p'} m_{p'}} \, d_{m_{p'}}(\mathrm{e}^{-u_{p'}})$$
$$\times (-1)^{m_p} \, \mathrm{e}^{-u_p/2(m_p+2n+1)} \, \mathrm{e}^{-\mathrm{i}\gamma_p(m_p+1)} d_{m_p}^{(n)}(\mathrm{e}^{-u_p}) \tag{C.11}$$

where in the first product a is identified with $\mathrm{e}^{-u_{p'}/2+(\mathrm{i}/\hbar)S_{p'}-\mathrm{i}\gamma_{p'}}$ and x with $\mathrm{e}^{-u_{p'}}$ $(p'\neq p)$, while in the second product $a = \mathrm{e}^{-u_p/2+(\mathrm{i}/\hbar)S_p-\mathrm{i}\gamma_p}$ and $x = \mathrm{e}^{-u_p}$. The values of $m_{p'}$ and m_p are selected so that

$$\mathcal{T}_\mu = \sum_{p'\neq p} m_{p'} T_{p'} + (m_p+1) T_p . \tag{C.12}$$

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